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16. Abstract <p>The boundary integral method was applied to the elastoplastic analysis of the torsion of prismatic bars, and the results are compared with those obtained by the finite difference method. Although fewer unknowns were used, very good accuracy was obtained with the boundary integral method. Both simply and multiply connected bodies can be handled with equal ease.</p>					
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SOLUTION OF ELASTOPLASTIC TORSION PROBLEM BY BOUNDARY INTEGRAL METHOD

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SUMMARY

The boundary integral method was applied to the elastoplastic analysis of the torsion of prismatic bars. The boundary of the bar cross section was divided into n intervals, and the unknown warping function was assumed constant over each interval. Since the unknowns of the problem appear only on the boundary, the dimensions of the problem are reduced effectively by one.

Numerical results were obtained for a bar with a square cross section, and comparisons were made with results obtained by the finite difference method. Very good accuracy was obtained using relatively small sets of linear algebraic equations. The convergence rate for the elastoplastic problem was also much faster using the boundary integral method than using the finite difference method. This should be reflected in an appreciable savings in computer time. The method can be readily programmed for a digital computer for both simply and multiply connected bodies.

INTRODUCTION

The problem of elastic torsion of prismatic bars is one of the classical problems of mechanics and as such has received extensive treatment. With the advent of the high speed digital computer the corresponding elastoplastic problem has also come to the forefront.

The usual solutions to the elastoplastic torsion problem assume perfect plasticity and are very often limited to cases in which the complete cross section is plastic, because the elastoplastic boundary is considered difficult to find. Recently, general solutions have been presented by the author (refs. 1 and 2) using finite differences and the method of elastic solutions and by Herakovitch (refs. 3 and 4) using a minimum rate principle of plasticity in conjunction with nonlinear programming. In reference 4

Herakovich compares his results with those of reference 2 and concludes that the minimum rate principle would require a great many more finite elements for its implementation in order to approach the accuracy of the finite difference method.

More recently a great deal of interest has been aroused by a new approach involving boundary integral techniques to solve elastic and elastoplastic problems. A general review of the boundary integral techniques is presented in reference 5. The first numerical application of these techniques to an elastoplastic problem involving a notched beam in bending is described in reference 6. A major advantage of these techniques is that a problem need only be solved for the unknowns on the boundary of the cross section of the body. This effectively reduces the dimensions of the problem by one.

The development of the theory of the boundary integral method to the torsion problem is described in general terms in reference 5. The purpose of the present report is to show specifically how the theory is applied to the elastoplastic torsion of bars of any geometry. Numerical computations are presented for bars having square cross sections. Comparisons are made for the elastic case with the known analytical solution and for the elastoplastic case with the finite difference solution.

METHOD OF ANALYSIS

The elastoplastic torsion problem can be formulated in several ways as shown in reference 5. In particular, for example, it is shown in reference 5 that by formulating the problem in terms of a stress function a closed form solution can be obtained for a circular bar with linear strain hardening. In general, however, it would seem that a formulation in terms of the warping function should be preferable, since the warping function (axial displacement) is physically more meaningful than the stress function and, more importantly, the distinction between simply connected and multiply connected regions disappears (ref. 7). The warping function formulation was therefore used herein.

Basic Equations

The basic equations necessary for the formulation to be used are as follows (see fig. 1 for the coordinate system):

Equilibrium equation:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad (1)$$

where τ_{xz} and τ_{yz} are the shear stresses acting on the cross section.
Stress-strain relations:

$$\left. \begin{aligned} \tau_{xz} &= 2G(\epsilon_{xz} - \epsilon_{xz}^p) \\ \tau_{yz} &= 2G(\epsilon_{yz} - \epsilon_{yz}^p) \end{aligned} \right\} \quad (2)$$

where G is the shear modulus, ϵ_{xz} and ϵ_{yz} are the total (tensor) strains, and ϵ_{xz}^p and ϵ_{yz}^p are the plastic shear strains. It should be noted that equation (2) implies using the deformation theory of plasticity. However, as shown in reference 8, both the total and incremental theories of plasticity furnish the same solution to the torsion problem provided either the cross section is circular or the material is perfectly plastic. It is reasonable to assume, therefore, that this will be approximately true for most practical problems. Indeed, it has been shown by Huth in reference 9 that for the case of a square cross section with strain hardening there is little difference between incremental and deformation theories. In what follows, therefore, use will be made of the deformation or total theories of plasticity, and the load will be assumed to be applied in one step. The use of incremental theories does not appreciably complicate the problem, and the necessary formulation is given in reference 2 for those desiring to use it.

Saint-Venant relations:

$$\left. \begin{aligned} \epsilon_{xz} &= \frac{1}{2} \left(-\alpha y + \frac{\partial w}{\partial x} \right) \\ \epsilon_{yz} &= \frac{1}{2} \left(\alpha x + \frac{\partial w}{\partial y} \right) \end{aligned} \right\} \quad (3)$$

where α is the angle of twist per unit length and w the warping function (the axial displacement).

Substituting equation (3) into equation (2) and the resulting equation into equation (1) result in

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = F(x, y) \quad (4a)$$

where

$$F(x, y) = 2 \left(\frac{\partial \epsilon_{xz}^p}{\partial x} + \frac{\partial \epsilon_{yz}^p}{\partial y} \right) \quad (4b)$$

The plastic strains appearing in the definition of the function $F(x, y)$ are of course in turn nonlinear functions of warping function w . They can be determined from (ref. 1)

$$\left. \begin{aligned} \epsilon_{xz}^p &= \frac{\epsilon_p}{\epsilon_{et}} \epsilon_{xz} \\ \epsilon_{yz}^p &= \frac{\epsilon_p}{\epsilon_{et}} \epsilon_{yz} \end{aligned} \right\} \quad (5)$$

where

$$\epsilon_{et} = \frac{2}{\sqrt{3}} \sqrt{(\epsilon_{xz})^2 + (\epsilon_{yz})^2} \quad (6a)$$

and

$$\epsilon_p = f(\epsilon_{et}) \quad (6b)$$

Equation (6b) represents the uniaxial stress-strain curve in terms of equivalent plastic strain against equivalent total strain; that is,

$$\epsilon_p = \epsilon_{et} - \frac{1}{3} \frac{\sigma_e}{G} \quad (6c)$$

where σ_e , the equivalent stress, represents the stress on the uniaxial stress-strain curve and ϵ_p the plastic strain on that curve. Thus, for a given stress-strain curve, the relation between ϵ_p and ϵ_{et} represented by equation (6b) can be determined using equation (6c). For the case of linear strain hardening, the relation (6b) can be written as

$$\epsilon_p = \frac{\epsilon_{et} - \frac{2}{3} (1 + \mu) \frac{\sigma_0}{E}}{1 + \frac{2}{3} (1 + \mu) \frac{m}{1 - m}} \quad (6d)$$

where σ_0 is the yield stress, μ Poisson's ratio, and m the linear strain hardening parameter (i.e., the ratio of the slope of the linear strain hardening line to the elastic modulus).

Boundary Conditions

The boundary conditions for an unloaded lateral surface are given by

$$l\tau_{xz} + m\tau_{yz} = 0 \quad (7)$$

where l and m are the direction cosines of the external normal to the surface with respect to the x and y axes, respectively. Substituting equations (2) and (3) into equation (7) and making use of the relation

$$l \frac{\partial w}{\partial x} + m \frac{\partial w}{\partial y} = \frac{\partial w}{\partial n}$$

result in

$$\frac{\partial w}{\partial n} = w' = \alpha(l y - m x) + 2(l \epsilon_{xz}^p + m \epsilon_{yz}^p) \quad (8)$$

where w' is the outward normal derivative of the warping function w at the boundary of the surface. For a rectangular boundary parallel to one of the coordinate axes, the second term on the right side of equation (8) vanishes. This follows from the fact that if $l = 0$, then $\epsilon_{yz}^p = 0$, and if $m = 0$, then $\epsilon_{xz}^p = 0$.

Boundary Integral Formulation

The differential equation (4a) can be readily converted to a boundary integral equation by using Green's second theorem as shown in reference 5. The resultant equation is

$$w(p) = \frac{1}{\pi} \iint_R F(Q) \ln r_{pQ} dA + \frac{1}{\pi} \int_C w(q) \frac{\partial}{\partial n_q} (\ln r_{pq}) dq - \frac{1}{\pi} \int_C w'(q) \ln r_{pq} dq \quad (9)$$

The notation in equation (9) is fully described in figure 2. The capital letters P and Q denote interior points while the lower case letters p and q denote boundary points. Thus, for example, r_{pQ} represents the distance between the boundary point p and the interior point Q . The outward normal derivative at the boundary point q is $\partial/\partial n_q$. The same equation applies for a point P in the interior of the region R except that the right side of the equation is divided by 2; that is,

$$w(P) = \frac{1}{2\pi} \int_R F(Q) \ln r_{PQ} dA + \frac{1}{2\pi} \int_C w(q) \frac{\partial}{\partial n_q} (\ln r_{Pq}) dq - \frac{1}{2\pi} \int_C w'(q) \ln r_{Pq} dq \quad (10)$$

In equation (9), the normal derivative $w'(q)$ is known from equation (8). If we assume that the function F defined by equation (4b) is also known, then equation (9) represents an integral equation for the unknown function $w(q)$ on the boundary C of the region R . Once this integral equation is solved for $w(q)$, the warping function w can be calculated at any interior point P by direct quadrature using equation (10).

Actually, however, the function $F(Q)$ is not known to start with, since it is a function of the plastic strains which are determined as part of the solution. However, $F(Q)$ can be determined by an iterative process. First, equation (9) is solved for an arbitrary distribution (say zero) of $F(Q)$. Then $w(P)$ is calculated in the interior of the region by means of equation (10). The total strains are computed from equation (3), and the plastic strains are computed by means of equations (5) and (6). A better approximation to the function F is then obtained from equation (4b). Equation (9) is now resolved, and the process is repeated until convergence is obtained.

Numerical Procedure

To solve equation (9) for the unknown function $w(p)$, the straightforward procedure of replacing the integrals by summations can be used. The boundary is divided into n intervals with a nodal point taken at the center of each interval. The unknown function is assumed constant over each interval. Similarly, the region R is divided into a number of cells and the function F assumed constant over each cell. Equation (9) is then written for each nodal point as follows:

$$\sum_{j=1}^n (a_{ij} - \delta_{ij}\pi) w_j = \sum_{j=1}^n b_{ij} w'_j + R_i \quad i = 1, 2, \dots, n \quad (11)$$

where δ_{ij} is the Kronecker delta (equal to 0 when $i \neq j$ and equal to 1 when $i = j$). The coefficients a_{ij} , b_{ij} , and R_i are given in the appendix. We thus have n equations for the n unknowns w_j . This set of equations can readily be solved by any standard procedure.

Once the w_j are known on the boundary, equation (10) can be used to calculate w at any interior point. However, as can be seen from equation (3), although w is not really needed to calculate the strains, its derivatives are. To determine the derivatives of w , equation (10) can be differentiated directly to give

$$\begin{aligned} \frac{\partial w(\mathbf{P})}{\partial x} = & \frac{1}{2\pi} \iint_R F(\mathbf{Q}) \frac{x_P - x_Q}{r_{PQ}^2} dx_Q dy_Q \\ & + \frac{1}{2\pi} \int_C w(q) \frac{[(x_P - x_q)^2 - (y_P - y_q)^2]l_q + 2(x_P - x_q)(y_P - y_q)m_q}{r_{Pq}^4} dq \\ & - \frac{1}{2\pi} \int_C w'(q) \frac{x_P - x_q}{r_{Pq}^2} dq \end{aligned} \quad (12)$$

For $\partial w(\mathbf{P})/\partial y$ we interchange x and y .

Again we replace the integrals by sums and write

$$\frac{\partial w(x_i, y_i)}{\partial x} = \frac{1}{2\pi} \left[\sum_{k=1}^n (w_k l_k A_{ijk} + w_k m_k B_{ijk} - w'_k C_{ijk}) + \sum_{k,l} F_{kl} D_{ijkl} \right] \quad (13)$$

where the coefficients A_{ijk} , B_{ijk} , C_{ijk} , and D_{ijkl} are listed in the appendix, l_k and m_k are the direction cosines of the normal at the k^{th} boundary interval, and $\sum_{k,l}$ is the sum for all the plastic cells in the region.

When the derivatives of w are used, the total strains are computed from equation (3), the plastic strains from equation (5) and (6), and the function F from equation (4b). The process is then repeated until convergence is obtained. The degree of convergence can be determined from a relation of the form

$$\frac{1}{M} \sum_{i=1}^M \left| \epsilon_x^p(P_i)_k - \epsilon_x^p(P_i)_{k-1} \right| < K \quad (14)$$

where M is the total number of points P_i flowing plastically and $k-1$ and k are two successive iterations. The convergence criterion K can be made as small as desired. For the calculations reported herein values of 10^{-4} and 10^{-5} were used for K .

For a cross section symmetric about one or both of the coordinate axes, advantage can be taken of the symmetries (or antisymmetries) to reduce the number of unknowns. For example, for the square plate for which calculations are reported herein, symmetry exists not only about the x and y axes, but also about the diagonals. Therefore, it is necessary to consider only the boundary between A and B (fig. 3) since the values of w are the same along AB, CD, EF, and GH, and these are the negatives of the values along CB, ED, GF, and AH. Similarly, the function F , defined by equation (4b), is antisymmetric with respect to both the x and y axes.

RESULTS AND DISCUSSION

Calculations were performed by the previous technique for a bar of square cross section as shown in figure 3. The dimensionless angle of twist per unit length β , defined as $\alpha a / \epsilon_0$ (where ϵ_0 is the yield strain and a is $1/2$ the side of the square), was increased in steps of one from $\beta = 1$ to $\beta = 6$. Linear strain hardening was assumed with values of the strain hardening parameter taken as 0 (perfect plasticity), 0.05, 0.1, and 0.2. Poisson's ratio was assumed as 0.3 in all calculations.

For $\beta = 1$, the bar is elastic and a comparison was made between the analytical solution as given, for example, in reference 7 as well as with the finite difference solution of reference 2. The results are shown in tables I to III. Table I shows the warping function as computed on the boundary of the bar cross section. The comparison with the analytical solution of reference 7 shows very good agreement with just 4 unknowns to solve for in the boundary integral method. The analytical solution of reference 7 uses an infinite series of which 30 terms were summed. Summing 50 terms did not change the answers to 5 significant figures. The warping function was not computed in reference 2.

Table II shows the comparison for the maximum shear stress (at the center of the edge of the square) and the moment with the analytical solution of reference 7 and the finite difference solution of reference 2. Again it is seen that with just 4 unknowns in the boundary integral method very good results are obtained, as good as the results obtained for the finite difference method using 55 unknowns.

Table III presents the dimensionless shear stress distribution in the x-direction ($\tau_x \equiv \tau_{xz}/2G\epsilon_0$) throughout the cross section using 10 unknowns for the boundary integral method and 55 unknowns for the finite difference method. Again excellent agreement was obtained. Actually, the results with 4 unknowns using the boundary integral method are almost as good, but the results with 10 unknowns are presented to match the actual (x, y) values of the finite difference results without having to cross plot.

The dimensionless angle of twist per unit length β was then increased in unit steps to a maximum value of $\beta = 6$ for each value of the strain hardening parameter m . The total boundary was divided into 80 intervals resulting in 10 equations for 10 unknowns. Several test calculations were made with fewer intervals, and the results indicated that using 48 intervals (6 unknowns) changed the moment and maximum stress by at most one in the third significant figure and changed the maximum plastic strain by about 3 percent. All the subsequent results are therefore shown for 80 intervals (10 unknowns), although from an engineering viewpoint 48 or even 32 intervals would be sufficient.

The results of the calculations are summarized in table IV and figures 4 to 6. Figure 4 shows the dimensionless moment defined as $M^* \equiv M/(2G\epsilon_0 a^3)$ for various values of β and m . Figure 5 shows the corresponding dimensionless maximum shear stresses defined as

$$\tau_{\max} = \frac{\tau}{2G\epsilon_0}$$

and figure 6 shows the spread of the plastic zones with an increase of the angle of twist β .

In all the calculations the convergence number K of equation (14) was taken either as 0.0001 or 0.00001. In many of the calculations both numbers were used in turn. The differences in the results were found to be insignificant. For example, the number of iterations for convergence for the case of maximum plastic flow, which occurred for $\beta = 6$ and $m = 0$, was 39 for $K = 0.0001$ and 53 for $K = 0.00001$, and the results were all the same to at least three significant figures. For the case $\beta = 5$ and $m = 0$, the number of iterations for $K = 0.0001$ was 33. For the same case using finite differences, 203 iterations were required.

CONCLUDING REMARKS

The boundary integral method was found to be very suitable for the elastoplastic analysis of the torsion of prismatic bars. Very good accuracy can be obtained by using relatively small sets of linear algebraic equations.

A comparison with the finite difference method indicates a great savings in the number of unknowns that have to be determined and also a much faster convergence rate using the method of successive elastic solutions for both formulations. This should be reflected in appreciable savings in computer time.

The boundary integral method can readily be programmed in a straightforward manner for a digital computer. The use of the warping function to formulate the problem permits applying the method with equal ease to both simply connected and multiply connected bodies.

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APPENDIX - BOUNDARY INTEGRAL COEFFICIENTS

The division of the boundary into intervals with their corresponding nodal points is shown in figure 7. The x and y coordinates of a boundary nodal point p_i are designated as (x_{bi}, y_{bi}) . The coordinates at the beginning and end of an interval (say interval j) are designated by (ξ_j, η_j) at the beginning of the interval and by (ξ_{j+1}, η_{j+1}) at the end of the interval. The interval lengths h_j need not be equal. The coordinates of the centroid of an interior cell where plastic flow occurs are designated by (x_k, x_l) .

The coefficients in equation (11) are then given by

$$\left. \begin{aligned} a_{ij} &= \int_{q_{j-(1/2)}}^{q_{j+(1/2)}} \frac{\partial}{\partial n_q} (\ln r_{piq}) dq \\ &\cong h_j \frac{(x_{bj} - x_{bi})l_j + (y_{bj} - y_{bi})m_j}{r_{ij}^2} \quad j \neq i \\ a_{ii} &= \pi - \sum_{k \neq i} a_{ik} \end{aligned} \right\} \quad (A1)$$

The last relation follows from the Gaussian condition (ref. 10); that is,

$$\int_C \frac{\partial}{\partial n_q} \ln r_{pq} dq = \pi$$

To evaluate the b_{ij} coefficients Simpson's rule is used for the case $i \neq j$ and closed form integration is used for the case $i = j$ since the integrand is singular for $i = j$. The result is

$$\left. \begin{aligned} b_{ij} &= \frac{h_j}{6} \ln \left[r_{i,j-(1/2)} + 4 \ln r_{ij} + \ln r_{i,j+(1/2)} \right] \quad i \neq j \\ b_{ii} &= h_i \left(\ln \frac{h_i}{2} - 1 \right) \end{aligned} \right\} \quad (A2)$$

$$R_i = -\frac{1}{4} \sum_{k,l} F_{kl} \ln r_{ikl} \Delta A_{kl} \quad (A3)$$

where $\sum_{k,l}$ is the sum for all the plastic cells in the region and ΔA_{kl} is the area of the cell with coordinates (x_k, y_l) .

The coefficients A_{ijk} , B_{ijk} , C_{ijk} , and D_{ijkl} are given as follows using Simpson's rule:

$$A_{ijk} = \frac{h_k}{6} \left\{ \frac{(x_i - \xi_k)^2 - (y_j - \eta_k)^2}{r_{ij,k-(1/2)}^4} + 4 \frac{(x_i - x_{bk})^2 - (y_j - y_{bk})^2}{r_{ijk}^4} + \frac{(x_i - \xi_{k+1})^2 - (y_j - \eta_{k+1})^2}{r_{ij,k+(1/2)}^4} \right\} \quad (A4)$$

$$B_{ijk} = \frac{h_k}{3} \left\{ \frac{(x_i - \xi_k)(y_j - \eta_k)}{r_{ij,k-(1/2)}^4} + 4 \frac{(x_i - x_{bk})(y_j - y_{bk})}{r_{ijk}^4} + \frac{(x_i - \xi_{k+1})(y_j - \eta_{k+1})}{r_{ij,k+(1/2)}^4} \right\} \quad (A5)$$

$$C_{ijk} = \frac{h_k}{6} \left[\frac{x_i - \xi_k}{r_{ij,k-(1/2)}^2} + 4 \frac{x_i - x_{bk}}{r_{ijk}^2} + \frac{x_i - \xi_{k+1}}{r_{ij,k+(1/2)}^2} \right] \quad (A6)$$

$$D_{ijkl} = \frac{x_i - x_k}{(x_i - x_k)^2 + (y_j - y_l)^2} \quad (A7)$$

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TABLE I. - COMPARISON OF VALUES OF DIMENSIONLESS WARPING FUNCTION ON BOUNDARY
OF ELASTIC SQUARE PLATE WITH EXACT ELASTIC SOLUTION

Boundary value, y	Exact warping function, w	Value of warping function by boundary integral method				Boundary value, y	Exact warping function, w	Value of warping function by boundary integral method			
		Intervals, n						Intervals, n			
		4	8	12	16			4	8	12	16
0.03125	0.01095	0.04311	0.02184	0.01459	0.01095	0.4313	0.1424	0.1446	0.1446	0.1433	0.1424
.04167	.01459					.5417	.1433				
.0625	.02185					.5625	.1448				
.09375	.03264					.5938	.1461				
.1250	.04328					.6250	.1461				
.1563	.05374			.05372	.6563	.1449					
.1875	.06390		.06386		.6875	.1422	.1419				
.2083	.07050			.07051	.7083	.1396			.1395		
.2188	.07380				.7188	.1380				.1380	
.2813	.09235				.09234	.7813	.1244				.1244
.2917	.09527			.09525	.7917	.1214			.1212		
.3125	.1009		.1009		.8125	.1148		.1140			
.3438	.1090			.1090	.8438	.1029				.1027	
.3750	.1165	.1159		.1164	.8750	.08864	.08826		.08811		
.4063	.1233			.1232	.9063	.07166				.07129	
.4375	.1293		.1292		.9375	.05169		.05142			
.4583	.1329			.1329	.9583	.03644			.03621		
.4688	.1346			.1346	.9688	.02808				.02796	

TABLE II. - COMPARISON OF ELASTIC SOLUTIONS FOR MAXIMUM
DIMENSIONLESS SHEAR STRESS AND DIMENSIONLESS
MOMENT FOR SQUARE BAR

	Exact solution	Finite difference method (55 eqs.)	Boundary integral method	
			Intervals, n	
			4	8
Dimensionless moment, M^*	1.125	1.122	1.128	1.127
Dimensionless maximum shear stress, τ_{\max}	.6754	.6725	.6724	.6747

TABLE III. - COMPARISON OF ELASTIC SOLUTIONS

FOR DIMENSIONLESS x DIRECTIONAL SHEARSTRESS DISTRIBUTION τ_x

[First number, exact; second, boundary integral method; third, finite difference method.]

y	x					
	0	0.2	0.4	0.6	0.8	1.0
	Elastic solutions					
1.0	0.675	0.658	0.605	0.507	0.342	0
	.675	.658	.605	.506	.339	↓
	.671	.654	.600	.500	.330	↓
0.8	0.492	0.476	0.427	0.338	0.198	0
	.492	.476	.427	.338	.199	↓
	.492	.476	.428	.339	.200	↓
0.6	0.339	0.326	0.287	0.219	0.121	0
	.339	.326	.287	.220	.122	↓
	.340	.327	.288	.220	.123	↓
0.4	0.212	0.203	0.176	0.132	0.0714	0
	.212	.203	.177	.132	.0717	↓
	.212	.203	.177	.133	.0720	↓
0.2	0.101	0.0971	0.0839	0.0623	0.0333	0
	.101	.0971	.0840	.0624	.0335	↓
	.102	.0973	.0842	.0626	.0336	↓

TABLE IV. - SUMMARY OF RESULTS FOR TORSION
OF SQUARE PRISMATIC BAR

Linear strain-hardening parameter, m	Dimensionless angle of twist per unit length, β	Dimensionless moment, M^*	Maximum dimensionless shear stress, τ_{\max}	Maximum dimensionless strain, $\gamma_{\max}^p = \frac{\gamma_{\max}^p}{\epsilon_0}$
0	2	1.81	0.751	0.792
	3	1.96	↓	1.70
	4	2.02		2.57
	5	2.06		3.42
	6	2.08		4.25
0.05	2	1.83	0.784	0.758
	3	2.04	.822	1.57
	4	2.15	.852	2.35
	5	2.24	.893	3.12
	6	2.31	.926	3.85
0.10	2	1.85	0.817	0.687
	3	2.11	.890	1.45
	4	2.27	.960	2.17
	5	2.41	1.03	2.86
	6	2.53	1.09	3.52
0.20	2	1.90	0.879	0.593
	3	2.25	1.02	1.24
	4	2.51	1.15	1.85
	5	2.74	1.28	2.43
	6	2.95	1.40	3.00

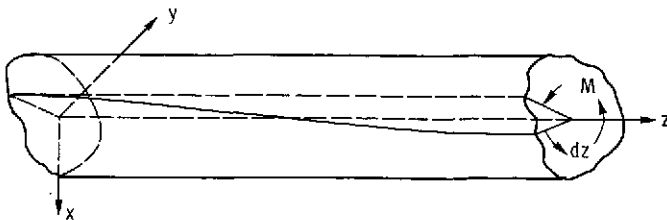


Figure 1. - Prismatic bar subject to twisting couple.

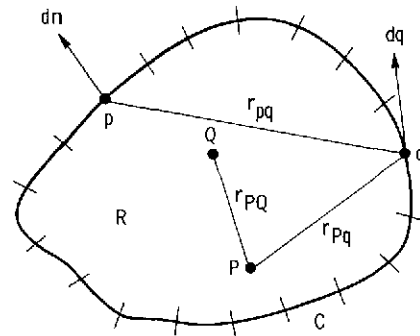


Figure 2. - Region R, boundary curve c, and geometric quantities entering into boundary integrals.

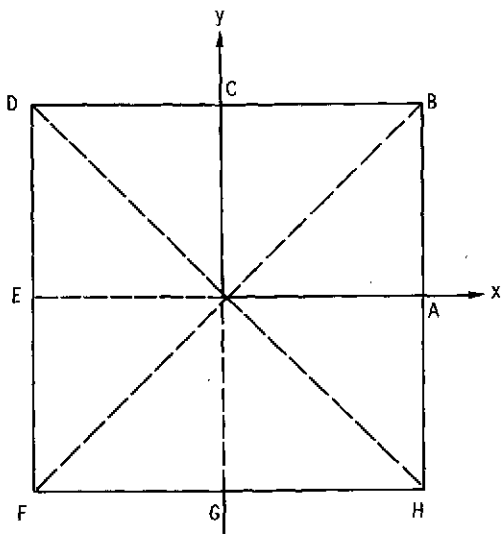


Figure 3. - Square cross section.

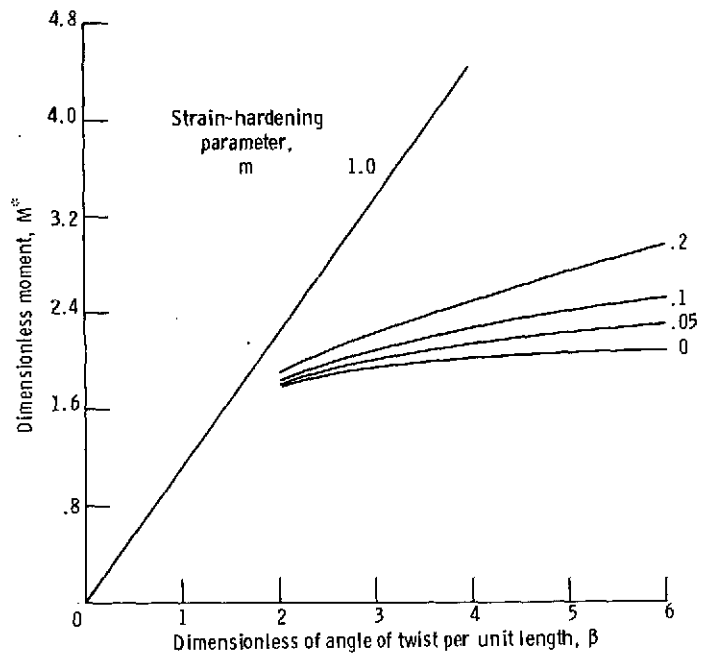


Figure 4. - Variation of dimensionless moment with dimensionless angle of twist per unit length for several values of strain-hardening parameter for square cross section.

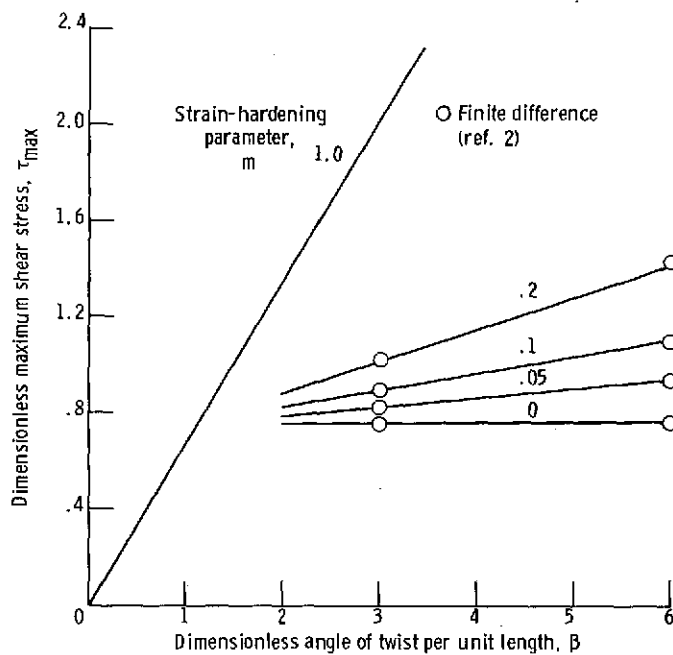


Figure 5. - Variation of dimensionless maximum shear stress with dimensionless angle of twist per unit length for several values of strain-hardening parameter for square cross section.

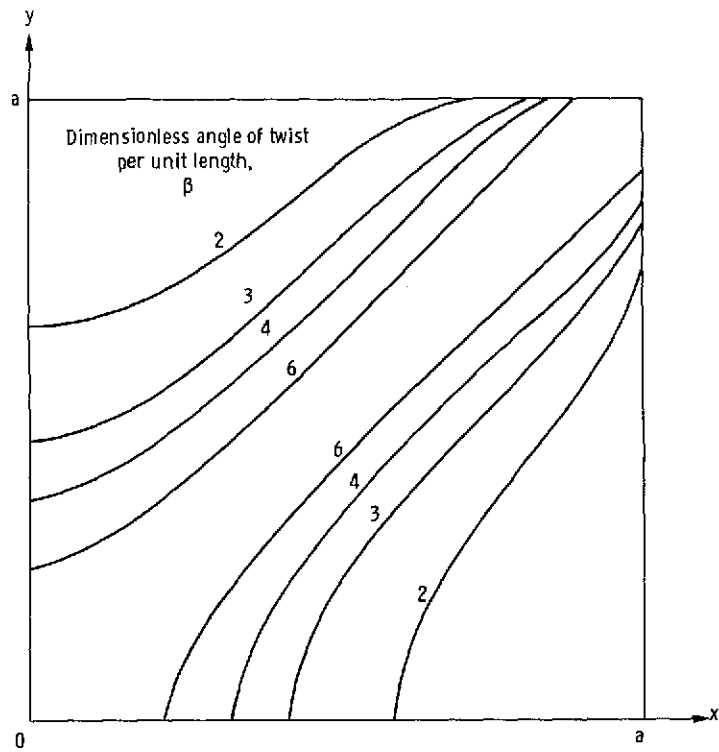


Figure 6. - Plastic zone boundaries in quadrant of square cross section as function of dimensionless angle of twist per unit length for strain-hardening parameter, 0.1.

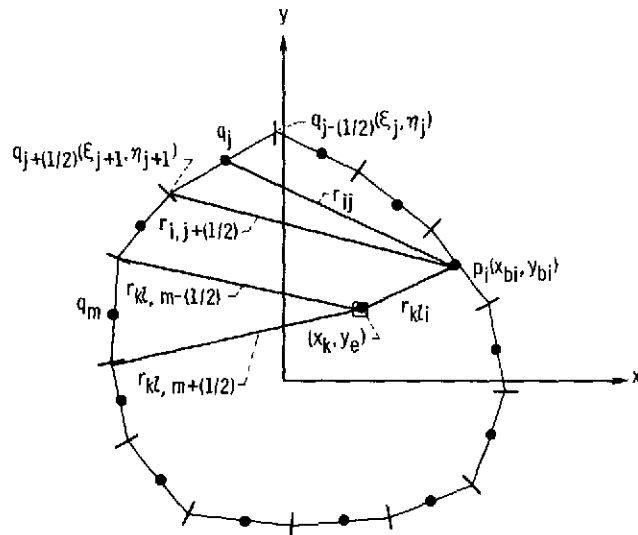


Figure 7. - Boundary and interior rotation for computing coefficients given in appendix.